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Are $N = 1$ and $N = 2$ supersymmetric quantum mechanics equivalent?

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Abstract

After recalling different formulations of the definition of supersymmetric quantum mechanics given in the literature, we discuss the relationships between them in order to provide an answer to the question raised in the title.

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1. Introduction

Supersymmetric quantum mechanics (SUSYQM) was introduced more than two decades ago by Nicolai [1] and Witten [2]. In its simplest form, it consists of the study of quantum mechanical systems which are described by a Hamiltonian operator of the form $H = \mathbb{Q}^2$ acting on a Hilbert space \mathcal{H} which admits a \mathbb{Z}_2 -grading, i.e., \mathcal{H} has the form of a direct sum: $\mathcal{H} = \mathcal{H}_b \oplus \mathcal{H}_f$.

The aim of the present paper is to elucidate the precise relationship between different formulations of the definition for SUSYQM which have been considered in the literature [1–4]. To start with, we will list these different defining relations as well as the results to be established in the present work.

2. Definitions and summary of results

The common starting point is a *quantum mechanical system* (\mathcal{H}, H) characterized by a self-adjoint operator $H \neq 0$ (the *Hamiltonian* or *energy* of the system) acting on a complex separable Hilbert space \mathcal{H} (the *state space*). As usual, the commutator and anticommutator of two operators A and B are denoted by $[A, B]$ and $\{A, B\}$, respectively. All operators to be considered are linear and the adjoint of an operator A is denoted by A^\dagger . Our concern will primarily be of an algebraic nature and we leave it to the mathematically minded reader to supplement the relevant analytical details such as domains of definition for operators, proper characterization of the anti-commutativity for unbounded self-adjoint operators, etc [5].

2.1. Definitions

Definition 1. The quantum mechanical system (\mathcal{H}, H) is called supersymmetric if there exists a finite number of self-adjoint operators Q_1, \dots, Q_N on \mathcal{H} such that

$$\{Q_i, Q_j\} = 2\delta_{ij}H \quad \text{for } i, j \in \{1, \dots, N\}. \quad (1)$$

The operators Q_1, \dots, Q_N are called supercharges (or supersymmetry generators).

From relations (1), it follows that the supercharges are conserved, i.e., they commute with the Hamiltonian: $[H, Q_i] = 0$ for $i \in \{1, \dots, N\}$. Since the latter relation also means that the Hamiltonian is invariant under the transformations generated by the Q_i , the operator H is called a *supersymmetric Hamiltonian* or *super-Hamiltonian*. With supersymmetric field theories [6] in mind, the algebra (1) with N supercharges is usually qualified as *N -extended supersymmetry algebra*.

Definition 2. The quantum mechanical system (\mathcal{H}, H) is called supersymmetric if there exists a finite number of non-self-adjoint operators q_1, \dots, q_M on \mathcal{H} such that

$$\{q_i, q_j^\dagger\} = 2\delta_{ij}H \quad \{q_i, q_j\} = 0 \quad \text{for } i, j \in \{1, \dots, M\}. \quad (2)$$

The operators q_1, \dots, q_M are called complex supercharges.

It follows that $\{q_i^\dagger, q_j^\dagger\} = 0$ and that $[H, q_i] = 0 = [H, q_i^\dagger]$ for $i, j \in \{1, \dots, M\}$. Note that the operators q_1, \dots, q_M satisfying (2) cannot be self-adjoint, otherwise $H = 0$ (contrary to our assumption $H \neq 0$).

Definition 3. The quantum mechanical system (\mathcal{H}, H) is said to be supersymmetric if there exists a finite number of self-adjoint operators Q_1, \dots, Q_n (called supercharges) as well as a bounded self-adjoint operator K (called involution), all of which operators act on \mathcal{H} and satisfy

$$K^2 = \mathbb{1} \quad \{K, Q_i\} = 0 \quad \text{for } i \in \{1, \dots, n\} \quad (3)$$

and

$$\{Q_i, Q_j\} = 2\delta_{ij}H \quad \text{for } i, j \in \{1, \dots, n\}. \quad (4)$$

The operator K is also referred to as the *Klein operator*, *chirality operator*, *fermion number operator*, *Witten parity operator* or *\mathbf{Z}_2 -grading operator*. Since K is a bounded operator, it can be (and it will be) assumed to be defined on the entire space \mathcal{H} . Note that the choice $K = \pm\mathbb{1}$ implies the trivial solution $Q_1 = \dots = Q_n = H = 0$ that we excluded. Definition 3 also implies $[H, Q_i] = 0$ for all values of i .

Definition 4. The present definition (involving m complex supercharges $\mathbf{q}_1, \dots, \mathbf{q}_m$) is the same as definition 2 supplemented with an involution K .

Obviously, definitions 3 and 4 are nothing but definitions 1 and 2, respectively, supplemented with the operator K . The crucial question is whether the existence of this operator already follows from the existence and properties of the supercharges, i.e., whether K necessarily represents a function of the supercharges or whether it is an extra independent input.

In our study of the relationship between the given definitions we will concentrate on the most popular special cases which we now summarize for later reference.

2.2. The most important special cases

The simplest and most studied supersymmetric systems are, respectively, given by the following cases:

- $N = 2$ in definition 1, i.e.,

$$Q_1^2 = Q_2^2 = H \quad \{Q_1, Q_2\} = 0. \tag{5}$$

- $M = 1$ in definition 2, i.e., with the notation $q \equiv q_1$:

$$\{q, q^\dagger\} = 2H \quad q^2 = 0. \tag{6}$$

- $n = 1$ in definition 3, i.e., with the notation $\mathbb{Q} \equiv \mathbb{Q}_1$:

$$K^2 = \mathbb{1} \quad \{K, \mathbb{Q}\} = 0 \quad \mathbb{Q}^2 = H. \tag{7}$$

- $n = 2$ in definition 3, i.e.,

$$K^2 = \mathbb{1} \quad \{K, \mathbb{Q}_1\} = \{K, \mathbb{Q}_2\} = 0 \quad \mathbb{Q}_1^2 = \mathbb{Q}_2^2 = H \quad \{\mathbb{Q}_1, \mathbb{Q}_2\} = 0. \tag{8}$$

- $m = 1$ in definition 4, i.e., one has a non-self-adjoint operator \mathbf{q} satisfying

$$K^2 = \mathbb{1} \quad \{K, \mathbf{q}\} = 0 \quad \{\mathbf{q}, \mathbf{q}^\dagger\} = 2H \quad \mathbf{q}^2 = 0. \tag{9}$$

2.3. Summary of results

For the sake of clarity, we summarize the relationships between the special cases (5)–(9) which are going to be established in what follows:

- (A) By writing $q = \frac{1}{\sqrt{2}}(Q_1 + iQ_2)$ and $\mathbf{q} = \frac{1}{\sqrt{2}}(\mathbb{Q}_1 + i\mathbb{Q}_2)$, one checks the equivalence of (5) and (6) and the equivalence of (8) and (9).
- (B) Relations (7) imply that the operator $\mathbb{Q}' = \pm iK\mathbb{Q}$ represents a second supercharge, i.e., remarkably enough, $n = 1$ implies $n = 2$.
- (C) The converse of (B) also holds, i.e., the two supercharges defining a $n = 2$ supersymmetric system are related by $\mathbb{Q}_2 = \pm iK\mathbb{Q}_1$.
- (D) From relations (5), one can deduce the existence of an involution operator K such that relations (8), or equivalently (7) or (9), hold.
- (E) From relations (9), one concludes that K , \mathbf{q} and H have the general form

$$K = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix} \quad \mathbf{q} = \sqrt{2} \begin{bmatrix} 0 & A^\dagger \\ 0 & 0 \end{bmatrix} \quad H = \begin{bmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{bmatrix} \tag{10}$$

where A is a linear operator. In the literature, these expressions for the complex supercharge and for the associated Hamiltonian (eventually with a specific choice of A in terms of the operators of position and momentum) are referred to as *Witten's model* of SUSYQM.

By combining the previous results, we conclude that the sets of relations (5)–(9) are equivalent and that every supersymmetric Hamiltonian satisfying any one of these sets of relations can be cast into the form (10). In other words, the latter expressions do not simply describe a specific model of SUSYQM (with one complex or two real supercharges), but they represent its most general form¹.

The proof of statements (C) and (E) will be provided in section 3.2.1 and that of (D) in section 4. The result (B) readily follows from the properties of the involution K [4]. The proof of (A) is as follows. Note that $q = \frac{1}{\sqrt{2}}(Q_1 + iQ_2)$ implies $q^\dagger = \frac{1}{\sqrt{2}}(Q_1 - iQ_2)$ and that these

¹ To be more precise, (10) is the most general form up to redefinitions of the supercharges leaving the Hamiltonian invariant—see section 2.4 below. In particular, application of unitary transformations may lead to more complicated expressions for the operators.

expressions are equivalent to $Q_1 = \frac{1}{\sqrt{2}}(q + q^\dagger)$, $Q_2 = \frac{-i}{\sqrt{2}}(q - q^\dagger)$. From

$$0 = q^2 = \frac{1}{2}(Q_1^2 - Q_2^2 + i\{Q_1, Q_2\}) \iff Q_1^2 = Q_2^2 \quad \text{and} \quad \{Q_1, Q_2\} = 0$$

$$2H = \{q, q^\dagger\} = Q_1^2 + Q_2^2$$

one concludes that $H = Q_1^2 = Q_2^2$ with $\{Q_1, Q_2\} = 0$ and vice versa.

2.4. Non-uniqueness of supercharges and extra symmetries

Suppose (\mathcal{H}, H) represents a supersymmetric system in the sense of equation (5) with supercharges Q_1 and Q_2 . Then, H can be expressed in a completely symmetric way in terms of the supercharges: $H = \frac{1}{2}(Q_1^2 + Q_2^2)$. From this expression, it is clear that the charges Q_1 and Q_2 are not unique: the reparametrization $Q'_i = \sum_{j=1}^2 a_{ij} Q_j$, where the matrix $A \equiv (a_{ij})$ describes a real orthogonal transformation (i.e., $A \in \mathcal{O}(2)$), leaves the defining relations of the supersymmetric system and, in particular, the Hamiltonian invariant. Thus, Q'_1 and Q'_2 represent an equivalent collection of supercharges for the given supersymmetric system and a supersymmetric Hamiltonian admits a larger invariance than supersymmetry since it is automatically *invariant under a rotation in the (Q_1, Q_2) -space*.

Similarly, in equation (8), the supercharges \mathbb{Q}_1 and \mathbb{Q}_2 can be transformed by a matrix $A \in \mathcal{O}(2)$ and, in (6) or (9), the complex supercharge can be changed by a phase factor $\lambda \in \mathcal{U}(1)$.

3. Consequences of the definitions

In this section, we show that definition 3 with $n = 1$, i.e., equations (7), implies the characteristic features that are generally associated with SUSYQM. Thereafter, we discuss the general form of such supersymmetric systems and we illustrate the results by two simple examples.

3.1. Characteristic features of supersymmetric systems

We outline the consequences of definition 3 for a single supercharge $\mathbb{Q}_1 \equiv \mathbb{Q}$ by expanding on the brief discussion presented in [4]. The inner product on the Hilbert space \mathcal{H} will be denoted by $\langle \cdot, \cdot \rangle$ and the induced norm by $\|\cdot\|$. As usual the restriction of an operator A on \mathcal{H} to a subspace $\mathcal{D} \subset \mathcal{H}$ is written as $A \upharpoonright \mathcal{D}$.

Since \mathbb{Q} is self-adjoint and $H = \mathbb{Q}^2$, we have $H \geq 0$ by virtue of

$$\langle \varphi, H\varphi \rangle = \langle \mathbb{Q}\varphi, \mathbb{Q}\varphi \rangle = \|\mathbb{Q}\varphi\|^2 \geq 0 \quad \text{for any } \varphi \in \mathcal{H}.$$

Thus, a supersymmetric Hamiltonian necessarily has a *non-negative spectrum*. As mentioned already, $H = \mathbb{Q}^2$ implies $[H, \mathbb{Q}] = 0$.

From $K^2 = \mathbb{1}$, it follows that the involution K only admits ± 1 as eigenvalues. Henceforth, K induces a direct sum decomposition of the Hilbert space \mathcal{H} : if $\varphi \in \mathcal{H}$, then

$$\begin{aligned} \varphi &= \frac{1}{2}(\varphi + K\varphi) + \frac{1}{2}(\varphi - K\varphi) \\ &\equiv \varphi_b + \varphi_f. \end{aligned} \tag{11}$$

In other words,

$$\mathcal{H} = \mathcal{H}_b \oplus \mathcal{H}_f \quad \text{with} \quad \begin{cases} \mathcal{H}_b = \{\varphi \in \mathcal{H} \mid K\varphi = +\varphi\} \\ \mathcal{H}_f = \{\varphi \in \mathcal{H} \mid K\varphi = -\varphi\}. \end{cases} \tag{12}$$

Since $K \neq \pm \mathbb{1}$, the subspaces \mathcal{H}_b and \mathcal{H}_f are non-trivial, i.e., different from \mathcal{H} and $\{0\}$. Motivated by the role played by the operator \mathbb{Q} in particle physics, the vectors belonging

to \mathcal{H}_b and \mathcal{H}_f are called, respectively, *bosonic* (or *even*) and *fermionic* (or *odd*) vectors². In the present context, this terminology only expresses the dichotomy introduced into the theory by the involution K : the precise physical interpretation depends on the example under consideration.

It is convenient to introduce a matrix notation for the vectors belonging to the direct sum (12): rather than writing $\varphi = (\varphi_b, 0) + (0, \varphi_f)$, we will use the matrix notation

$$\varphi = \begin{bmatrix} \varphi_b \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \varphi_f \end{bmatrix} = \begin{bmatrix} \varphi_b \\ \varphi_f \end{bmatrix}.$$

With this notation for the vectors, the operator K reads

$$K = \begin{bmatrix} \mathbb{1}_b & 0 \\ 0 & -\mathbb{1}_f \end{bmatrix} \tag{13}$$

where $\mathbb{1}_b$ denotes the restriction of the identity operator to the subspace \mathcal{H}_b of \mathcal{H} , and analogously for $\mathbb{1}_f$.

The involution K not only induces a decomposition of the state space \mathcal{H} , but also of the algebra of operators acting on \mathcal{H} . In fact, let

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

denote a generic operator acting on $\mathcal{H} = \mathcal{H}_b \oplus \mathcal{H}_f$. Then

$$[K, M] = 0 \iff M = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \tag{14}$$

and

$$\{K, M\} = 0 \iff M = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}. \tag{15}$$

In analogy with the terminology introduced for the state vectors, the operators commuting with the involution K are called *bosonic* or *even operators* while those anticommuting with K are referred to as *fermionic* or *odd operators*.

Since \mathbb{Q} is self-adjoint and anticommutes with K , the result (15) applied to $M = \mathbb{Q}$ implies that

$$\mathbb{Q} = \begin{bmatrix} 0 & A^\dagger \\ A & 0 \end{bmatrix} \tag{16}$$

where A is a linear operator. Let us now apply \mathbb{Q} to a vector $\varphi \in \mathcal{H}$:

$$\mathbb{Q}\varphi = \begin{bmatrix} 0 & A^\dagger \\ A & 0 \end{bmatrix} \begin{bmatrix} \varphi_b \\ \varphi_f \end{bmatrix} = \begin{bmatrix} A^\dagger\varphi_f \\ A\varphi_b \end{bmatrix}.$$

Since the resulting vector again belongs to the space $\mathcal{H}_b \oplus \mathcal{H}_f$, we have

$$\mathbb{Q} : \mathcal{H}_b \rightarrow \mathcal{H}_f \quad \mathbb{Q} : \mathcal{H}_f \rightarrow \mathcal{H}_b \tag{17}$$

which means that \mathbb{Q} *exchanges bosonic and fermionic states*. It is precisely this fundamental property of \mathbb{Q} which is at the origin of the terminology ‘supersymmetry’ operator.

By virtue of $H = \mathbb{Q}^2$ and (16), the Hamiltonian H has the form

$$H = \begin{bmatrix} A^\dagger A & 0 \\ 0 & A A^\dagger \end{bmatrix} \equiv \begin{bmatrix} H_+ & 0 \\ 0 & H_- \end{bmatrix} \tag{18}$$

with $H_+ : \mathcal{H}_b \rightarrow \mathcal{H}_b$ and $H_- : \mathcal{H}_f \rightarrow \mathcal{H}_f$.

² One also says that \mathcal{H} is a \mathbf{Z}_2 -graded Hilbert space with a *fixed parity* [7].

We remark that expressions (13), (16) and (18) are known as the *standard* or *fermion number representation* of SUSYQM. Equivalent though more complicated expressions can be obtained by applying a unitary transformation to all of these operators.

To conclude, we come to the fundamental spectral property of every supersymmetric system. Suppose³

$$H\varphi = E\varphi \quad \text{with } E > 0.$$

By applying the operator \mathbb{Q} to this relation and using $[H, \mathbb{Q}] = 0$, we find

$$H(\mathbb{Q}\varphi) = E(\mathbb{Q}\varphi).$$

Hence, if φ is an eigenstate of the Hamiltonian H , then $\mathbb{Q}\varphi$ also represents an eigenstate of H associated with the same eigenvalue $E > 0$. (This argument is not valid for $E = 0$: the relation $H\varphi = 0$ infers $0 = \langle \varphi, H\varphi \rangle = \|\mathbb{Q}\varphi\|^2$, therefore $\mathbb{Q}\varphi = 0$ is the null vector which is not an eigenvector by definition.)

According to (17), $\varphi \in \mathcal{H}_b$ (respectively \mathcal{H}_f) implies $\mathbb{Q}\varphi \in \mathcal{H}_f$ (respectively \mathcal{H}_b). Thus, we have derived the following fundamental property of a quantum mechanical system which is supersymmetric in the sense of definition 3.

Theorem 3.1 (degeneracy structure of a supersymmetric system). *For an $n = 1$ supersymmetric system, the non-vanishing eigenvalues of the Hamiltonian admit the same numbers of bosonic and fermionic eigenvectors.*

In other words, the partner Hamiltonians $H \upharpoonright \mathcal{H}_b$ and $H \upharpoonright \mathcal{H}_f$ are *isospectral*, except possibly for the eigenvalue zero. For later reference, we recall that the difference between the numbers of bosonic and fermionic states of zero energy is known as the *Witten* or *supersymmetric index* of H [2, 4, 8]:

$$\begin{aligned} \text{ind}_S H &= \dim \ker [H \upharpoonright \mathcal{H}_b] - \dim \ker [H \upharpoonright \mathcal{H}_f] \\ &= \dim \ker A - \dim \ker A^\dagger. \end{aligned} \quad (19)$$

Here ‘ker’ denotes the kernel and A is the operator defining the supercharge \mathbb{Q} according to (16). We note that expression (19) is only well defined if \mathbb{Q} has some extra properties like being of Fredholm type, i.e., if the eigenvalue 0 of \mathbb{Q} has finite multiplicity [8].

To conclude, we note that in the physically or mathematically interesting applications, the Hilbert spaces \mathcal{H} , \mathcal{H}_b and \mathcal{H}_f are of infinite dimension so that we have the isomorphism

$$\mathcal{H}_b \oplus \mathcal{H}_f \cong \mathbf{C}^2 \otimes \mathcal{H}_{bf} \quad \text{with } \mathcal{H}_{bf} \cong \mathcal{H}_b \cong \mathcal{H}_f. \quad (20)$$

The involution K then takes the form

$$K = \sigma_3 \otimes \mathbb{1} \quad \text{with } \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (21)$$

which is simply written as $K = \sigma_3$ in most of the literature. In this setting, the (2×2) -matrix format of the supercharge (16) or of the involution (21) can also be expressed in terms of the so-called *fermionic creation and annihilation operators* which act on the Hilbert space \mathbf{C}^2 and satisfy canonical anticommutation relations: we have

$$\mathbb{Q} = f^\dagger \otimes A + f \otimes A^\dagger \quad \text{with } f = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (22)$$

³ Here, the real number E belongs to the discrete spectrum of H if φ is an element of the Hilbert space \mathcal{H} (or, more precisely, if it is an element of the domain $\mathcal{D}(H) \subset \mathcal{H}$ of the operator H). It belongs to the continuous spectrum of H if φ represents a weak (distributional) solution of the eigenvalue equation.

as well as $\sigma_3 = [f, f^\dagger]$, i.e.,

$$K = [f, f^\dagger] \otimes \mathbb{1} \tag{23}$$

where

$$\{f, f^\dagger\} = \mathbb{1}_2 \quad \{f, f\} = 0. \tag{24}$$

The fact that \mathbb{Q} is linear in the fermionic operators f and f^\dagger (acting on \mathbf{C}^2) reflects the fact that \mathbb{Q} is fermionic (odd) with respect to the \mathbf{Z}_2 -grading on \mathcal{H} .

3.2. General form of an $n = 1$ (or $n = 2$) supersymmetric system

3.2.1. Supercharges and involution. Let us consider an $n = 1$ supersymmetric system. By virtue of (16), the supercharge has the general form $\mathbb{Q} \equiv \mathbb{Q}_1 = \begin{bmatrix} 0 & A_1^\dagger \\ A_1 & 0 \end{bmatrix}$. We can decompose the operator A_1 according to $A_1 = a_1 + ia_2$ where the self-adjoint operators a_1 and a_2 represent the Hermitian and anti-Hermitian (real and imaginary) parts of A_1 . Thus, we have

$$\mathbb{Q}_1 = \begin{bmatrix} 0 & a_1 - ia_2 \\ a_1 + ia_2 & 0 \end{bmatrix}. \tag{25}$$

Given the operators a_1 and a_2 , one can find a second supercharge of the same form, $\mathbb{Q}_2 \equiv \begin{bmatrix} 0 & b_1 - ib_2 \\ b_1 + ib_2 & 0 \end{bmatrix}$, which is ‘normalized’ in the sense that $\mathbb{Q}_2^2 = \mathbb{Q}_1^2$ and which is ‘orthogonal’ to \mathbb{Q}_1 in the sense that $\{\mathbb{Q}_1, \mathbb{Q}_2\} = 0$: this supercharge is determined up to a global sign and given by

$$\mathbb{Q}_2 = \begin{bmatrix} 0 & -a_2 - ia_1 \\ -a_2 + ia_1 & 0 \end{bmatrix}. \tag{26}$$

In other words, $(b_1, b_2) = \pm(-a_2, a_1)$.

One immediately verifies that

$$\mathbb{Q}_2 = -iK\mathbb{Q}_1 \quad \text{or equivalently} \quad \mathbb{Q}_1 = +iK\mathbb{Q}_2 \tag{27}$$

hence we have the following general result: *every $n = 2$ supersymmetric system is of the form (27), i.e., $n = 1$ is equivalent to $n = 2$.* The associated *complex supercharge* takes the simple and well-known form which was put forward in equation (10):

$$\mathbf{q} \equiv \frac{1}{\sqrt{2}}(\mathbb{Q}_1 + i\mathbb{Q}_2) = \sqrt{2} \begin{bmatrix} 0 & A_1^\dagger \\ 0 & 0 \end{bmatrix}. \tag{28}$$

If \mathcal{H} , \mathcal{H}_b and \mathcal{H}_f are of infinite dimension, the operator A_1 acts on $\mathcal{H}_{bf} \cong \mathcal{H}_b \cong \mathcal{H}_f$ and we can rewrite expressions (25), (26) as

$$\mathbb{Q}_1 = \sigma_1 \otimes a_1 + \sigma_2 \otimes a_2, \quad \mathbb{Q}_2 = \sigma_2 \otimes a_1 - \sigma_1 \otimes a_2. \tag{29}$$

Here, σ_1 and σ_2 are the Pauli matrices which represent a basis of complex Hermitian (2×2)-matrices anticommuting with σ_3 . (Note that the operators (29) anticommute with $K = \sigma_3 \otimes \mathbb{1}$.) These Hermitian matrices generate the Clifford algebra associated with the Euclidean metric in a two-dimensional vector space,

$$\{\sigma_\alpha, \sigma_\beta\} = 2\delta_{\alpha\beta} \mathbb{1} \quad (\alpha, \beta = 1, 2). \tag{30}$$

Following the practice of quantum mechanics, we can combine the generators σ_1 and σ_2 satisfying relation (30) into a fermionic annihilation operator

$$f \equiv \frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{31}$$

the latter acting on the Hilbert space \mathbf{C}^2 and satisfying the canonical anticommutation relations (24). The general form of supercharges for an $n = 2$ supersymmetric system, as given by (29), then reads

$$\mathbb{Q}_i = f^\dagger \otimes A_i + f \otimes A_i^\dagger \quad \text{with} \quad A_1 \equiv a_1 + ia_2, A_2 \equiv -a_2 + ia_1. \quad (32)$$

The associated complex supercharge is given by $\mathbf{q} = \sqrt{2}f \otimes A_1^\dagger$ and the *Hamiltonian* $H = \mathbb{Q}_1^2 = \mathbb{Q}_2^2$ takes the form

$$\begin{aligned} H &= \{f, f^\dagger\} \otimes (a_1^2 + a_2^2) + [f, f^\dagger] \otimes i[a_1, a_2] \\ &= \mathbb{1}_2 \otimes (a_1^2 + a_2^2) + \sigma_3 \otimes i[a_1, a_2]. \end{aligned} \quad (33)$$

3.2.2. Examples. Many interesting Hamiltonians are supersymmetric, e.g. see [1–4]. The prototype examples are the spin- $\frac{1}{2}$ particle in a one-dimensional superpotential or in a constant two-dimensional magnetic field. A simple and an important example of a more mathematical nature is the Laplace–Beltrami operator (acting on the Hilbert space of the differential forms defined on a Riemannian manifold): slight modifications of this example have been used to prove deep mathematical theorems [2, 4]. By way of illustration, we now elaborate briefly on a quantum mechanical system whose supersymmetric nature is not very familiar.

The free particle in one dimension. As pointed out quite recently [9], the free particle moving on a line represents the simplest example of SUSYQM. In this case, the involution operator is realized [10] by the *parity operator*⁴:

$$(K\varphi)(x) = \varphi(-x) \quad \text{for} \quad \varphi \in \mathcal{H} = L^2(\mathbf{R}). \quad (34)$$

Indeed, this operator is bounded, self-adjoint and satisfies $K^2 = \mathbb{1}$. Since the momentum operator $p \equiv p_x$ changes sign under a parity transformation, $KpK^\dagger = -p$, we have $\{K, p\} = 0$. Henceforth, the operator $\mathbb{Q} = \frac{1}{\sqrt{2}}p$ represents a supercharge for this quantum mechanical system:

$$H = \mathbb{Q}^2 = \frac{1}{2}p^2 \quad \{K, \mathbb{Q}\} = 0. \quad (35)$$

Let us verify that all of these expressions admit the standard matrix representation. The decomposition (11) into bosonic and fermionic vectors is now to be interpreted as a decomposition into even and odd parity functions:

$$\begin{aligned} \varphi(x) &= \frac{1}{2}[\varphi(x) + \varphi(-x)] + \frac{1}{2}[\varphi(x) - \varphi(-x)] \\ &\equiv \varphi_+(x) + \varphi_-(x). \end{aligned}$$

The momentum operator modifies the parity,

$$\begin{aligned} p\varphi &= (p\varphi)_+ + (p\varphi)_- = \frac{1}{2}(\mathbb{1} + K)p\varphi + \frac{1}{2}(\mathbb{1} - K)p\varphi \\ &= p\frac{1}{2}(\mathbb{1} - K)\varphi + p\frac{1}{2}(\mathbb{1} + K)\varphi \\ &= p(\varphi_-) + p(\varphi_+) \end{aligned}$$

i.e. $(p\varphi)_\pm = p(\varphi_\mp)$.

Let us introduce the projection operators $\Pi_\pm = \frac{1}{2}(\mathbb{1} \pm K)$ which satisfy $\Pi_+\Pi_- = \Pi_-\Pi_+ = 0$ and $\Pi_+ + \Pi_- = \mathbb{1}$. Since $\{K, p\} = 0$, we obtain

$$p = \mathbb{1}p\mathbb{1} = \left. \begin{aligned} &\Pi_-p\Pi_+ + \Pi_+p\Pi_- \\ &= p \upharpoonright \mathcal{H}_+ + p \upharpoonright \mathcal{H}_- \end{aligned} \right\} \quad \text{with} \quad \left\{ \begin{aligned} &\Pi_-p\Pi_+ : \mathcal{H}_+ \rightarrow \mathcal{H}_- \\ &\Pi_+p\Pi_- : \mathcal{H}_- \rightarrow \mathcal{H}_+ \end{aligned} \right. \quad (36)$$

⁴ The author of [10] refers to this choice as the ‘minimally bosonized SUSYQM’ and discusses the one-dimensional particle in a parity-odd superpotential.

and $(\Pi_- p \Pi_+)^{\dagger} = \Pi_+ p \Pi_-$. For the sake of clarity, we presently put a hat on vectors and operators when referring to the matrix expressions,

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_+ \oplus \mathcal{H}_- \ni \begin{bmatrix} \varphi_+ \\ \varphi_- \end{bmatrix} \equiv \hat{\varphi} & \hat{K} &= \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix} \\ \sqrt{2}\hat{Q} &= \hat{p} \equiv \begin{bmatrix} 0 & 0 \\ p_{-+} & 0 \end{bmatrix} + \begin{bmatrix} 0 & p_{+-} \\ 0 & 0 \end{bmatrix} \\ \hat{H} &= \hat{Q}^2 = \frac{1}{2} \begin{bmatrix} p_{+-}p_{-+} & 0 \\ 0 & p_{-+}p_{+-} \end{bmatrix} \end{aligned} \tag{37}$$

where the two contributions of \hat{p} correspond to those of p displayed in (36).

In this example, the spectrum of the superpartners $H_+ \equiv H \upharpoonright \mathcal{H}_+$ and $H_- \equiv H \upharpoonright \mathcal{H}_-$ is purely continuous and the generalized even and odd parity eigenfunctions associated with the spectral values $E = \frac{1}{2}\rho^2$ are given by

$$\begin{aligned} \varphi_{\rho}^{(+)}(x) &= \cos(\rho x) & \text{for } \rho \geq 0 \\ \varphi_{\rho}^{(-)}(x) &= \sin(\rho x) & \text{for } \rho > 0. \end{aligned} \tag{38}$$

Thus, $\rho = 0$ is a non-degenerate spectral value while the double degeneracy of all other spectral values is a manifestation of supersymmetry [9].

By virtue of the result (B) stated in section 2.3, a second supercharge exists and is given by $\mathbb{Q}_2 \equiv \pm iK\mathbb{Q} = \frac{\pm i}{\sqrt{2}}Kp$. However, this operator as well as the complex supercharge $\mathbf{q} = \frac{1}{\sqrt{2}}(\mathbb{Q} + i\mathbb{Q}_2)$ are non-local since they explicitly involve the parity operator. (The definition of locality that we have in mind here is the following one [14]. Let $C_0^{\infty}(\mathbf{R}) \subset L^2(\mathbf{R})$ denote the space of smooth functions $f : \mathbf{R} \rightarrow \mathbf{C}$ with compact support. An operator $A : C_0^{\infty}(\mathbf{R}) \rightarrow L^2(\mathbf{R})$ is said to be *local* if the support of Af is contained in the support of f for all $f \in C_0^{\infty}(\mathbf{R})$).

The spin- $\frac{1}{2}$ particle in a three-dimensional magnetic field. A less basic example is given by Pauli's Hamiltonian for a spin- $\frac{1}{2}$ particle in a magnetic field $\vec{B} = \text{rot } \vec{A}$. This operator acts on $\mathcal{H} = L^2(\mathbf{R}^3) \otimes \mathbf{C}^2$ and has the form

$$2H = (\vec{p} - \vec{A})^2 \mathbb{1}_2 - \vec{B} \cdot \vec{\sigma}. \tag{39}$$

Here, $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ and, for simplicity, we do not spell out the tensor product symbols in the present example. As was already noted in the early days of SUSYQM [11], we have

$$H = \mathbb{Q}^2 \quad \text{with} \quad \sqrt{2}\mathbb{Q} = (\vec{p} - \vec{A}) \cdot \vec{\sigma}$$

and $K\mathbb{Q}K^{-1} = -\mathbb{Q}$, where K represents the parity operator⁵. The latter equation can also be rewritten as $\{K, \mathbb{Q}\} = 0$ and therefore we actually have an $n = 1$ supersymmetric system. However, just as in the previous example, the second supercharge and the complex supercharge are given by non-local operators.

A simple matrix representation for the state vector $\phi = [\varphi, \psi]^t \in \mathcal{H}$ (where 't' denotes transposition) is defined by the 4-component column vector $\hat{\phi} = [\varphi_+, \psi_+, \varphi_-, \psi_-]^t$, where φ_+ and φ_- denote, respectively, the even and odd parity parts of $\varphi \in L^2(\mathbf{R}^3)$. The operators characterizing the supersymmetric system then read

$$\hat{K} = \begin{bmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{bmatrix} \quad \sqrt{2}\hat{Q} = \begin{bmatrix} 0 & (\vec{p} - \vec{A}) \cdot \vec{\sigma} \\ (\vec{p} - \vec{A}) \cdot \vec{\sigma} & 0 \end{bmatrix}$$

⁵ Note that $K\vec{\sigma}K^{-1} = \vec{\sigma}$ since $\vec{S} \equiv \frac{1}{2}\vec{\sigma}$ represents the angular momentum of spin.

$$\hat{H} = \hat{Q}^2 = \frac{1}{2} \begin{bmatrix} (\vec{p} - \vec{A})^2 \mathbb{1}_2 - \vec{B} \cdot \vec{\sigma} & 0 \\ 0 & (\vec{p} - \vec{A})^2 \mathbb{1}_2 - \vec{B} \cdot \vec{\sigma} \end{bmatrix}.$$

Here, we have suppressed the indices ‘+’ and ‘-’ denoting the restriction of operators to the subspaces \mathcal{H}_+ and \mathcal{H}_- , see equation (37).

4. Construction of an involution from two supercharges

In this section, we will deal with statement (D) made in section 2.3, i.e., we will show that one can construct an involution operator K from the two supercharges Q_1 and Q_2 defining an $N = 2$ supersymmetric system. To do so, we first try to find a concrete expression for the involution which is present in an $n = 2$ supersymmetric system.

As we have seen in section 3.2, the supercharges Q_1 and Q_2 defining an $n = 2$ system are related by

$$Q_2 = -iKQ_1 \quad (40)$$

(or $Q_2 = +iKQ_1$). Equation (40) can be solved for K ,

$$K = iQ_2Q_1^{-1} \quad \text{on} \quad (\ker Q_1)^\perp \quad (41)$$

where $(\ker Q_1)^\perp$ denotes the orthogonal complement of the subspace $\ker Q_1$.

In view of expression (41), we introduce an involution into the setting of definition 1 by defining

$$K = iQ_2Q_1^{-1} \quad \text{on} \quad (\ker Q_1)^\perp. \quad (42)$$

Clearly, the extension of the operator K to all of \mathcal{H} requires further discussion. Before dealing with this issue, we note that relation $Q_1^2 = Q_2^2$ for two self-adjoint operators Q_1 and Q_2 implies that the kernels of these operators coincide:

$$Q_1\varphi = 0 \iff Q_2\varphi = 0. \quad (43)$$

In fact, $Q_1\varphi = 0$ is equivalent to

$$0 = \|Q_1\varphi\|^2 = \langle Q_1\varphi, Q_1\varphi \rangle = \langle \varphi, Q_1^2\varphi \rangle = \langle \varphi, Q_2^2\varphi \rangle = \|Q_2\varphi\|^2$$

hence $Q_2\varphi = 0$.

Furthermore, for any self-adjoint operator Q , we have the equivalence

$$Q\varphi = 0 \iff Q^2\varphi = 0. \quad (44)$$

Indeed, the left-hand side obviously implies the right-hand side and the converse statement follows from

$$0 = \langle \varphi, Q^2\varphi \rangle = \langle Q\varphi, Q\varphi \rangle = \|Q\varphi\|^2.$$

Theorem. *Let (H, \mathcal{H}) be a supersymmetric system in the sense of definition 1 with $N = 2$, that is, assume that there exist self-adjoint operators Q_1 and Q_2 satisfying $Q_1^2 = Q_2^2 \equiv H$ and $\{Q_1, Q_2\} = 0$. Then, we have the following:*

- (i) *The operator K defined on $(\ker Q_1)^\perp$ by (42) admits an extension to all of \mathcal{H} which represents a non-trivial involution anticommuting with Q_1 and Q_2 . Hence, (H, \mathcal{H}) is an $n = 2$ supersymmetric system in the sense of definition 3.*

(ii) More specifically, if Q_1 has a kernel of finite dimension, then there exists a one-parameter family of extensions parametrized by the integer number $\text{ind}_S H \equiv d_+ - d_-$ where $d_+, d_- \in \{0, 1, 2, \dots\}$ are subject to the condition

$$d_+ + d_- = d \equiv \dim \ker Q_1. \tag{45}$$

Accordingly, there are $d + 1$ possible values for $\text{ind}_S H$:

$$\text{ind}_S H \in \{-d, -d + 2, \dots, d - 2, d\}.$$

Proof. The Hilbert space \mathcal{H} can be decomposed into a direct sum of the kernel of Q_1 (i.e., the eigenspace associated with the eigenvalue zero) and its orthogonal complement:

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_\perp \quad \text{with} \quad \mathcal{H}_0 = \ker Q_1 = \ker H \quad \mathcal{H}_\perp = (\ker Q_1)^\perp. \tag{46}$$

The operator $Q_1 \upharpoonright \mathcal{H}_\perp$ is invertible, hence $K \upharpoonright \mathcal{H}_\perp$ can be defined as in equation (42). On the space \mathcal{H}_\perp , the operator K satisfies

$$K^\dagger = K \quad K^2 = \mathbb{1} \quad \{K, Q_1\} = \{K, Q_2\} = 0. \tag{47}$$

The definition (42) of K on \mathcal{H}_\perp is equivalent to the relation

$$Q_2 = -iKQ_1 \quad \text{on} \quad \mathcal{H}_\perp \tag{48}$$

and the question is whether or not one can define an extension of the operator $K \upharpoonright \mathcal{H}_\perp$ to all of \mathcal{H} such that the relation $Q_2 = -iKQ_1$ and relations (47) also hold on \mathcal{H}_0 . The validity of $Q_2 = -iKQ_1$ on \mathcal{H}_0 is equivalent to

$$Q_2\varphi = -iKQ_1\varphi \quad \text{for all} \quad \varphi \in \mathcal{H}_0.$$

This equation holds trivially since both sides vanish by virtue of (43), whatever the expression of K . Thus, the only constraints for the definition of K on \mathcal{H}_0 consist of the conditions $K^\dagger = K$ and $K^2 = \mathbb{1}$. Operators with these properties exist and any one of them will be suitable for our theorem. In particular, if the eigenvalue zero of Q_1 is of finite multiplicity, then $K \upharpoonright \mathcal{H}_0$ is (up to unitary equivalence) a diagonal matrix with eigenvalues ± 1 , i.e.,

$$K \upharpoonright \mathcal{H}_0 = \text{diag}(1, \dots, 1; -1, \dots, -1)$$

with $d_+ \geq 0$ entries 1 and $d_- \geq 0$ entries -1 , subject to the condition (45). Since there is no constraint for the integer $d_+ - d_-$, any value satisfying $|d_+ - d_-| \leq d_+ + d_- \equiv d$ can be chosen: each one gives rise to another extension of K to all of \mathcal{H} . \square

5. Concluding remarks

Our discussion shows that a precise answer to the question raised in the title of our paper can only be given if one specifies the definition of SUSYQM that one has in mind. Obviously, $n = 1$ and $n = 2$ SUSYQM are equivalent. Furthermore, it seems that the examples presented in the literature for $N = 1$ SUSYQM (like Pauli's Hamiltonian) actually represent examples of $n = 1$ since an involution exists, though the second supercharge of the ensuing $n = 2$ system is non-local in this case.

There is no reason for a true $N = 1$ supersymmetric system (i.e., $H = Q_1^2$ with no involution K that satisfies $\{K, Q_1\} = 0$) to be equivalent to an $N = 2$ system⁶. For such a system, the property of being 'even' or 'odd' is not defined and therefore one cannot infer any of the typical properties associated with supersymmetric systems. Accordingly, the statement

⁶ A simple example for such a system is given by the supercharge $\sqrt{2}Q_1 = (\vec{p} - \vec{A}) \cdot \vec{\sigma} + \varphi \mathbb{1}_2$ where φ represents a superpotential that does not have a definite space parity.

made in the pioneering work [2] that ‘the simplest supersymmetric quantum mechanical system has $N = 2$ ’ should indeed be interpreted as stating that $N = 1$ SUSYQM is not truly supersymmetric.

From our discussion, we can conclude that the conceptually simplest approach to SUSYQM is the approach that starts with one supercharge \mathbb{Q} that anticommutes with an involution operator K . Simpler or physically more transparent expressions can eventually be obtained by using \mathbb{Q} and K to construct a second supercharge or by introducing a complex supercharge in terms of the latter two charges.

The line of arguments presented in our work can be generalized to a large extent to the case of SUSYQM with more than two supercharges, as well as to parasupersymmetric and fractional supersymmetric quantum mechanics. This discussion is beyond the scope of the present paper and will be reported elsewhere [12, 13].

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